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A Differential Equation for Surface Waves in Layers with Varying Thickness*

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The problems of elastic surface wave propagation in layers of nonuniform thickness have been considered to be prototypes for seismic wave propagation across continental margins and other regions of the earth of varying crustal thickness. In almost all of the previous work, approximate solutions have been obtained for particular idealized geometries [1]–[7]. However, in a previous paper [8], the propagation of Love waves in a single layer with nonuniform thickness has been studied by using a modified version of the "principle of localization" [9], with the purpose of computing phase velocities in the zone of nonuniform thickness. According to the principle of localization, it is assumed that the sloping boundary is made up of an infinite number of small steps and that the wave undergoes instantaneous transmission and reflection at each step; the reflection and transmission occur as if the layers to the left and to the right of the step were actually of uniform thickness, differing by an infinitesimally small amount. Although the method is applicable for arbitrary geometry of the layer, it depends crucially upon the solution of an auxiliary problem, namely, the determination of the transmission and reflection coefficients for waves propagating past a step. Although an exact solution to this auxiliary problem does not exist in the literature, an approximate solution has been given which makes the application of the solution practical. The phase shifts at any point are the results of an interference between the transmitted and the reflected waves.

Knopoff and Mal [8] have stated that the wave number for surface waves in a wedge-shaped structure can be expanded as a power series in the angle of the wedge, for small angles, as

$$k = k_0 + ik_1\theta + k_2\theta^2 + \cdots$$

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with all the coefficients k_j real. Thus the perturbation in the wave number, given by the real part of the above equation, is an even function of the wedge angle. Hence no anisotropy in the wave number for surface waves exists at a point in the wedge. The wave number is the same for waves propagating toward the vertex of the wedge and away from it. In this paper we concern ourselves with the construction of a general theory of eigenfunctions, for a completely arbitrary geometry of the layers, especially for the case in which the thicknesses of the layers change by a finite amount. The results of this calculation agree with the above assumption of isotropy. The above conclusion holds for surface waves in a two-dimensional multilayered medium with a free surface and with nonparallel interfaces. For simplicity, however, we consider a half-space with one overlying homogeneous layer; the theory in its present form also applies to multilayered media.

We assume that the layer is of uniform thickness H except in a certain finite region $0 < x_1 < a$ (Fig. 1), which we call the transition zone. We further assume that the surface waves are generated at the left extreme of the structure. Then the incident wave, that is the wave travelling to the right and

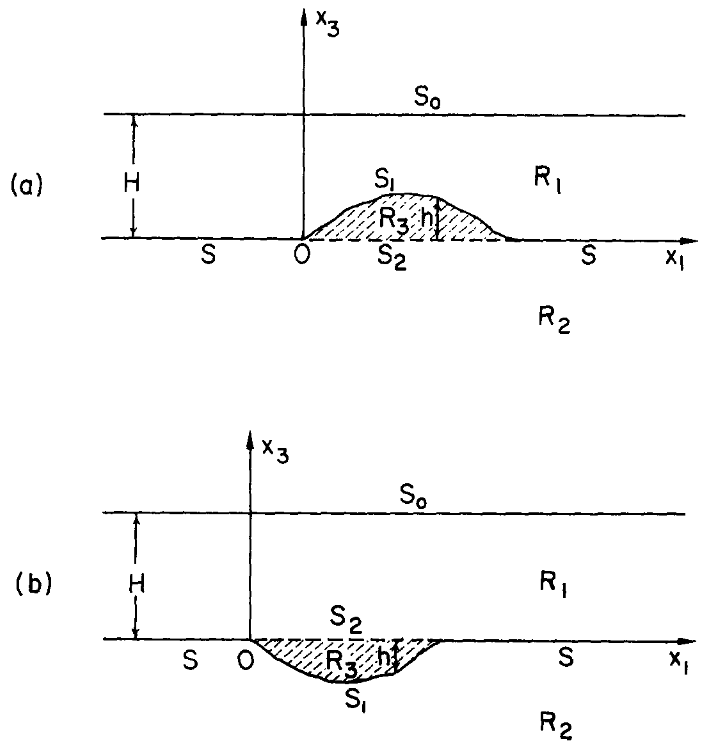

FIG. 1.

originating at $-\infty$, is the eigenfunction for a layered half-space with uniform layer thickness H. As the wave passes the transition zone, it undergoes reflection and transmission at every point. This results in a wave reflected to the left and one transmitted to the right of the transition zone. Outside the transition zone, these reflected and transmitted waves are again the eigenfunctions for the structure with uniform layer thickness. Thus the total wave motions outside the transition zone can be described completely in terms of a linear combination of the eigenfunction and its complex conjugate.

The situation is more complex in the transition zone. A cartesian coordinate system is inappropriate to describe the wave motion in this region. It is not possible to write down a solution in closed form which simultaneously satisfies the boundary conditions at the free surface and at the interface. As noted above, a precise mathematical description of the process by which surface waves undergo reflection and transmission at a sloping interface is not known. A convenient way of describing the effect of the region R_3 is to assume that every point in this region behaves as a secondary source emitting body and surface waves in all directions. Our purpose is to determine the strength of the sources as a function of the incident field and to describe the continuous reflection and transmission processes taking place in this region.

In this paper we solve the problem of Rayleigh waves incident upon such a discontinuity in structure. Let $e^{-i\omega t}\mathbf{u}^0(\mathbf{x})$, $e^{-i\omega t}\mathbf{v}^0(\mathbf{x})$ be the displacement vectors in the layer and the half space respectively due to the incident waves. Then $e^{-i\omega t}\mathbf{u}^0(\mathbf{x})$ satisfies the conditions of zero normal stress at the free surface and the conditions of continuity at the interface $x_3 = 0$ (Figure 1). These latter conditions are

$$\begin{gathered} \mathbf{u}^0 = \mathbf{v}^0 \\ \tau^0_{i3}(\mathbf{u}) = \sigma^0_{i3}(\mathbf{v}), \qquad i = 1, 3 \end{gathered} \tag{1}$$

where $\tau^0_{ij}(\mathbf{u})$ and $\sigma^0_{ij}(\mathbf{v})$ are the stresses due to the incident waves in the layer and the half-space, respectively. Let $e^{-i\omega t}\mathbf{u}(\mathbf{x})$ and $e^{-i\omega t}\mathbf{v}(\mathbf{x})$ denote the actual wave fields in the layer of nonuniform thickness and the half-space. We assume that R_3 is a region of inhomogeneity either in the layer (Fig. 1a) or the half-space (Fig. 1b). It has been shown [10] that in either of these cases the problem can be reconstructed in such a way that the inhomogeneity is replaced by a set of body forces distributed in R_3 and a prescribed jump in the normal derivatives of the displacements across the surface $S_1 + S_2$ bounding the region R_3. Thus,

$$\begin{aligned} u_k(\mathbf{x}) &= u_k{}^0(\mathbf{x}) + \int_{R_3} \mathscr{H}_k[\mathbf{x}, \boldsymbol{\xi}; \mathbf{v}(\boldsymbol{\xi})]\, d\boldsymbol{\xi}, \qquad \mathbf{x} \in R_1 \\ v_k(\mathbf{x}) &= v_k{}^0(\mathbf{x}) + \int_{R_3} \mathscr{H}_k[\mathbf{x}, \boldsymbol{\xi}; \mathbf{v}(\boldsymbol{\xi})]\, d\boldsymbol{\xi}, \qquad \mathbf{x} \in R_2 \\ \mathscr{H}_k[\mathbf{x}, \boldsymbol{\xi}; \mathbf{v}(\boldsymbol{\xi})] &= \omega^2 \Delta\rho G_{ki}(\boldsymbol{\xi}, \mathbf{x}) v_i(\boldsymbol{\xi}) - \Delta c_{ijpq} G_{ki,j} v_{p,q}(\boldsymbol{\xi}). \end{aligned} \tag{2}$$

In the above $e^{-i\omega t}G_{ki}(\boldsymbol{\xi}, \mathbf{x})$ is the ith component of the normalized displacement vector at $\boldsymbol{\xi}$ due to a point force at $\mathbf{x}$ acting in the k-direction in a single layered half-space with uniform layer thickness H, and

$$\begin{aligned} \Delta\rho &= \rho_2 - \rho_1 \\ \Delta c_{ijpq} &= (\lambda_2 - \lambda_1)\,\delta_{ij}\,\delta_{pq} + (\mu_2 - \mu_1)(\delta_{ip}\,\delta_{jq} + \delta_{iq}\,\delta_{jp}), \end{aligned} \tag{3}$$

where ρ_1, μ_1, λ_1 and ρ_2, μ_2, λ_2 are the densities and the elastic constants in the layer and the half-space respectively. A proof of Eq. (2) alternate to that given by Mal and Knopoff [10] is given in Appendix I.

We note that Eq. (2) is an exact integral representation of the field outside the region R_3. A usual procedure to evaluate (2) is to approximate the field inside R_3 by the incident field. However, this approximation is valid only if the volume of the region R_3 and the contrast between the elastic properties of the region R_3 and those of the surrounding material, are both very small [10]. We wish to consider the case where neither of these conditions need to be satisfied by our model. In the general case it is extremely difficult to determine the complete elastodynamic field inside R_3. However, for practical purposes, one simplification can be made which enables us to determine the field inside R_3 with comparative ease. This is a consequence of the fact that the Green's function $\mathbf{G}_k(\boldsymbol{\xi}, \mathbf{x})$ can be written as the sum of two terms, one giving the body wave contributions and the other giving the surface waves. Thus the integrals in (2) can be written as a sum of terms which describe the body and surface waves separately. Let us assume that it is possible to separate these two kinds of waves experimentally and that we focus our attention on the surface wave contributions to the above integrals. Thus we shall neglect the body wave parts of the Green's function. A further simplification is made by taking into account the fact that the surface wave contributions of the Green's function can be written entirely in terms of the eigenfunctions $\mathbf{u}^0(\mathbf{x})$ and $\mathbf{v}^0(\mathbf{x})$ [1].

Consider the structure described in Fig. 1a. Let

$$\begin{aligned} x_3 &= h(x_1),\ 0 < x_1 < a \\ &= 0,\ x > a \end{aligned} \tag{4}$$

be the equation of the interface. Thus R_3 is confined to the region $0 < x_3 < H$. Then Green's function at a point $\boldsymbol{\xi}$ in R_1 may be written as

$$\begin{aligned} \mathbf{G}_k(\boldsymbol{\xi}, \mathbf{x}) &= \frac{1}{R}\, u_k{}^0(\mathbf{x})\bar{\mathbf{u}}^0(\boldsymbol{\xi}), \qquad \xi_1 < x_1 \\ &= \frac{1}{R}\, \bar{u}_k{}^0(\mathbf{x})\mathbf{u}^0(\boldsymbol{\xi}), \qquad \xi_1 > x_1 \end{aligned} \tag{5}$$

for $0 < x_3 < H$, and

$$G_k(\boldsymbol{\xi}, \mathbf{x}) = \frac{1}{R} v_k^0(\mathbf{x})\bar{\mathbf{u}}^0(\boldsymbol{\xi}), \qquad \xi < x_1$$

$$= \frac{1}{R} \bar{v}_k^0(\mathbf{x})\mathbf{u}^0(\boldsymbol{\xi}), \qquad \xi_1 > x_1 \tag{6}$$

for $x_3 < 0$.

The normalization factor R is given by (Appendix II, (ii))

$$R = -2 \left[\int_0^H \bar{u}_i^0 \tau_{i1}^0(\mathbf{u})\, d\xi_3 + \int_{-\infty}^0 \bar{v}_i^0 \sigma_{i1}^0(\mathbf{v})\, d\xi_3 \right]. \tag{7}$$

We note that the Green's function given above does not have any discontinuity in its derivatives across the horizontal plane through the source.

By using (5) and (6), Eqs. (2) may be written as follows:

$$u_k(\mathbf{x}) = C(x_1)\, u_k^0(\mathbf{x}) + D(x_1)\, \bar{u}_k^0(\mathbf{x}),\ 0 < h < x_3 < H \tag{8a}$$

$$v_k(\mathbf{x}) = C(x_1)\, v_k^0(\mathbf{x}) + D(x_1)\, \bar{v}_k^0(\mathbf{x}),\ x_3 < 0, \tag{8b}$$

where $0 < x_1 < a$, and

$$P(\xi_1) = \int_0^{h(\xi_1)} L[\bar{\mathbf{u}}^0(\boldsymbol{\xi}), \mathbf{v}(\boldsymbol{\xi})]\, d\xi_3 \tag{9a}$$

$$Q(\xi_1) = \int_0^{h(\xi_1)} L[\mathbf{u}^0(\boldsymbol{\xi}), \mathbf{v}(\boldsymbol{\xi})]\, d\xi_3 \tag{9b}$$

$$L[\mathbf{u}, \mathbf{v}] = \omega^2 \Delta\rho u_i v_i - \Delta c_{ijpq} u_{i,j} v_{p,q} \tag{9c}$$

$$C(x_1) = 1 + \frac{1}{R} \int_0^{x_1} P(\xi_1)\, d\xi_1 \tag{9d}$$

$$D(x_1) = \frac{1}{R} \int_{x_1}^a Q(\xi_1)\, d\xi_1\, . \tag{9c}$$

For $x_1 > a$

$$u_k(\mathbf{x}) = C(a)\, u_k^0(\mathbf{x}) \tag{10a}$$

and for $x_1 < 0$

$$u_k(\mathbf{x}) = u_k^0(\mathbf{x}) + D(0)\, \bar{u}_k^0(\mathbf{x}). \tag{10b}$$

The displacement vector given by (8b) is valid only in the half-space $x_3 < 0$. However, it is also valid on the line $x_3 = 0, 0 < x_1 < a$ and $\mathbf{v}$ and $\mathbf{v}^0$ are analytic functions of x_1, x_3 in the region R_3. Thus, by the principle of analytic continuation, the representation (8b) is also valid in the region R_3. It should be noted that the displacement vector defined by (8a) and (8b) does not satisfy the continuity conditions on S_1. This is unavoidable and is a consequence of the fact that the body waves generated in the transition zone have been neglected. In other words, it is not possible to describe the complete elastodynamic field in the transition zone by means of surface waves only.

Let $f(x_1)$ and $g(x_1)$ be the normalized horizontal and vertical components of the displacement in the layer within the transitional zone:

$$\begin{aligned} f(x_1) &= e^{ik_0x_1}u_1(\mathbf{x})/u_1{}^0(\mathbf{x}) \\ g(x_1) &= e^{ik_0x_1}u_3(\mathbf{x})/u_3{}^0(\mathbf{x}). \end{aligned} \tag{11}$$

Then from Eq. (8a) we have

$$\begin{aligned} f(x_1) &= C(x_1)\, e^{ik_0x_1} + D(x_1)\, e^{-ik_0x_1} \\ g(x_1) &= C(x_1)\, e^{ik_0x_1} - D(x_1)\, e^{-ik_0x_1}, \end{aligned} \tag{12}$$

where we have assumed that the phase of the horizontal component is zero at $x_1 = 0$ and, by the demonstration of Appendix II, the phase of the vertical component differs from that of the horizontal component by $\pi/2$. Hence, from (12) the normalized field $e^{ik_0x_1}$ has added to it a component due to interaction with the changing structure to the left, with differential forward scattering coefficient $(1/R)\, P(\xi_1)\, d\xi_1$ and a component due to interaction with the structure to the right of x_1 with differential back scattering coefficient $(1/R)\, Q(\xi_1)\, d\xi_1$.

Substituting the expression (8b) for $\mathbf{v}(\mathbf{x})$ in (9a, b), we obtain the integral equations

$$\begin{aligned} P(x_1) = {} & C(x_1)P_0(x_1) + D(x_1)\bar{Q}_0(x_1) \\ & - \frac{1}{R}\{A_0(x_1)P(x_1) - \bar{B}_0(x_1)Q(x_1)\} \end{aligned} \tag{13a}$$

$$\begin{aligned} Q(x_1) = {} & C(x_1)Q_0(x_1) + D(x_1)\bar{P}_0(x_1) \\ & - \frac{1}{R}\{B_0(x_1)P(x_1) - \bar{A}_0(x_1)Q(x_1)\}, \end{aligned} \tag{13b}$$

where

$$\begin{aligned} P_0(\xi_1) &= \int_0^{h(\xi_1)} L[\bar{\mathbf{u}}^0(\boldsymbol{\xi}), \mathbf{v}^0(\boldsymbol{\xi})]\, d\xi_3 \\ Q_0(\xi_1) &= \int_0^{h(\xi_1)} L[\mathbf{u}^0(\boldsymbol{\xi}), \mathbf{v}^0(\boldsymbol{\xi})]\, d\xi_3 \end{aligned} \tag{14}$$

$$\begin{aligned} A_0(\xi_1) &= \Delta c_{ijp1} \int_0^{h(\xi_1)} v_p{}^0 \bar{u}^0_{i,j}(\boldsymbol{\xi})\, d\xi_3 \\ B_0(\xi_1) &= \Delta c_{ijp1} \int_0^{h(\xi_1)} v_p{}^0 u^0_{i,j}(\boldsymbol{\xi})\, d\xi_3 . \end{aligned} \tag{15}$$

By using (13) and the properties of the incident field derived in Appendix II, $f(x_1)$ and $g(x_1)$ can be shown to satisfy the following pair of coupled differential equations:

$$\begin{aligned} \frac{df}{dx_1} - i(k_0 - M)\, g(x_1) &= 0 \\ \frac{dg}{dx_1} - i(k_0 - N)\, f(x_1) &= 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} M &= (P_1 - Q_1)/(R_0 + A_1 - B_1) \\ N &= (P_1 + Q_1)/(R_0 + A_1 + B_1). \end{aligned} \tag{17}$$

The quantities P_1, Q_1, A_1, B_1 are real functions of h only and, along with R_0, are defined in Appendix II.

Equations (15) can be written in the form,

$$\frac{d}{dx}\left(\frac{1}{T}\frac{df}{dx}\right) + Sf(x) = 0 \tag{18a}$$

$$\frac{d}{dx}\left(\frac{1}{S}\frac{dg}{dx}\right) + Tg(x) = 0 \tag{18b}$$

with

$$\begin{aligned} T &= k_0 - M \\ S &= k_0 - N, \end{aligned} \tag{19}$$

and where we have dropped the suffix on x_1.

Equations (18) are exact differential equations describing the wave motion in the layer as long as the converted body waves and the higher mode surface waves can be neglected. Although the integral representations (2) are derived under the assumption that the layer is of uniform thickness H on both sides of the transitional zone, Eqs. (18) may be assumed to hold for more general geometries. One important case is where the layer has unequal uniform thickness on the two sides of the inhomogeneity. One must, however, be careful about the convergence of the solutions to (18) as $a \to \infty$.

Solutions to (18) are familiar. We may transform them for example, into the canonical form

$$\frac{d^2G}{dx^2} + k^2(x)G(x) = 0 \tag{20}$$

by the substitution

$$g(x) = \sqrt{S(x)}\,G(x), \tag{21}$$

where

$$k^2(x) = TS - \frac{1}{2}\left\{\frac{N''}{S} + \frac{3}{2}\left(\frac{N'}{S}\right)^2\right\}. \tag{22}$$

In the above, the primes denote differentiation with respect to x. Equation (20) is the reduced wave equation for propagation in a stratified medium and has been discussed in detail in the theory of acoustic and electromagnetic waves. It is to be noted that the local wave number $k(x)$ depends upon the local layer thickness as well as the slope and the curvature of the interface at the point x. The usual methods are available to solve (20) up to any desired degree of accuracy [11]. The solution can be written as a sum of contributions due to reflection and transmission at the sloping boundary.

The WKBJ solution to (20) depends only on the structure to the left of the point of observation, for waves incident from the left. For small slopes the "reflected" waves and the "transmitted" waves are of higher order than the lowest order solution,

$$G_0(\mathbf{x}) = \frac{1}{\sqrt{k(x)}}\, e^{i\int_0^x k(s)ds}. \tag{23}$$

Thus if $|h'(x)| \ll 1$ and sharp corners are not present in $h(x)$, the phase shift in the forward travelling waves between two points (x_1, H) and (x_2, H) on the free surface is given by,

$$\int_{x_1}^{x_2} k(s)\, ds, \tag{24}$$

where $k(x)$ is given by (22). Since (22) is of second order in the operation d/dx, to this order of approximation there is no anisotropy in the shift in wave number due to propagation "up" or "down" an inclined interface.

Similar expressions can be obtained for the displacements and the phase shifts for wave propagation across a thickening crust, for which the equation to the interface is (Fig. 1b),

$$\begin{aligned} x_3 &= -h(x_1),\ 0 < x_1 < a \\ &= 0,\ x_1 > a,\ x_1 < 0. \end{aligned} \tag{25}$$

In this case the inhomogeneity R_3 is in the half-space and the appropriate Green's function is given by

$$\begin{aligned} \mathbf{G}_k(\boldsymbol{\xi}, \mathbf{x}) &= \frac{1}{R}\, u_k{}^0(\mathbf{x})\bar{\mathbf{v}}^0(\boldsymbol{\xi}), \qquad \xi_1 < x_1 \\ &= \frac{1}{R}\, \bar{u}_k{}^0(\mathbf{x})\mathbf{v}^0(\boldsymbol{\xi}), \qquad \xi_1 > x_1 \end{aligned} \tag{26}$$

for $0 < x_3 < H$, and

$$
\begin{aligned}
\mathbf{G}_k(\boldsymbol{\xi}, \mathbf{x}) &= \frac{1}{R} v_k{}^0(\mathbf{x})\bar{\mathbf{v}}^0(\boldsymbol{\xi}), \qquad \xi_1 < x_1 \\
&= \frac{1}{R} \bar{v}_k{}^0(\mathbf{x})\mathbf{v}^0(\boldsymbol{\xi}), \qquad \xi_1 > x_1
\end{aligned} \tag{27}
$$

for $x_3 > H$. The representation (2) can now be used to obtain the differential equations for the eigenfunctions and the eigenvalues.

The theory can be extended to treat the case in which the layer is of uniform thicknesses H and H_1 to the left and right of the inhomogeneity, (Fig. 2a). For

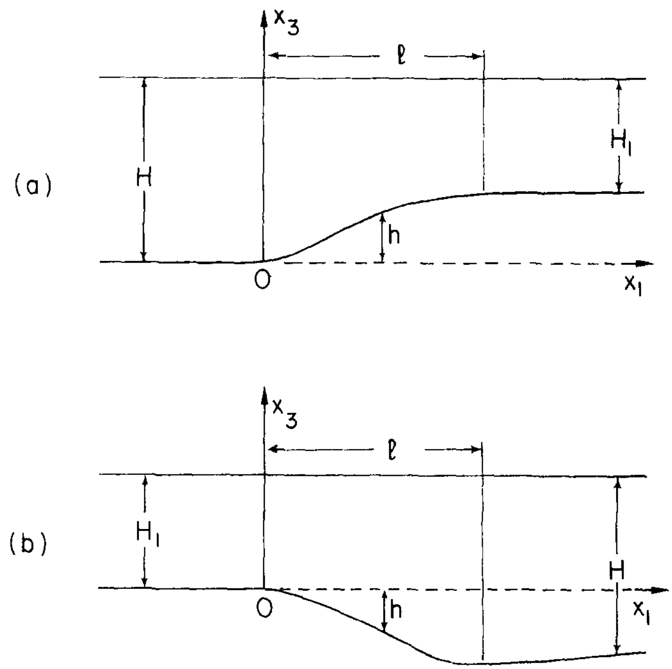

FIG. 2.

waves incident from the thicker side a formal solution can be obtained by making $a \to \infty$ in the results of Case I. It has been shown [9] that the solution converges if

$$
\begin{gathered}
|k(x)| < a_0 > 0, \qquad 0 < x < \infty \\
\int_0^l |k'(x)|\, dx = b_0 < 2a_0 .
\end{gathered} \tag{28}
$$

By choosing h and h' sufficiently small, $k(x)$ can be made to satisfy the above conditions.

For two points (x_1, H), (x_2, H) in the transition zone, the phase shift is

$$\int_{x_1}^{x_2} k(s)\, ds,$$

where $k(x)$ is given by (22).

We now consider two cases of the calculation of $k(x)$ in the light of the observation that the quantities P_1, Q_1, A_1, B_1, and R_0 are functions only of the constrast in properties and the height of the transition at the point x where the calculation of $k(x)$ is being made, as well as of the incident wave function. In Fig. 2a, consider the problem of the calculation of (22) at a point in the region where the structure to the right of the discontinuity is already horizontal and plane parallel. For a point of observation in the region where the thickness of the layer is everywhere H_1, the wave number is that appropriate to a layer of thickness H_1 with an interface of zero slope. In this region, $N' = N'' = 0$. Thus $\sqrt{TS}$ is the wave number appropriate for a plane layer of thickness H_1 and zero slope. It therefore follows that the leading term in the approximation (22) is the flat layer approximation; that is, in Fig. 1, to the lowest order of approximation, we can replace the inclined interface at every point with a horizontal layer having the local layer thickness.

We have derived in this paper a differential equation (18) for Rayleigh waves in a medium with sloping interfaces; this differential equation is exact as long as the body waves can be neglected. To the lowest order, the eigenvalues depend only upon the properties of the medium beneath the point $\mathbf{x}$, and upon the geometry at that point. The corrections to this lowest order approximation for the eigenvalues are isotropic with respect to the direction of propagation of the incident field. It must, however, be noted that the total wave motion at any point in the transition zone is given by an infinite series which, in general, cannot be decomposed into right and left travelling waves. Thus, although the perturbations in the eigenvalues are at most of the second order in the deviations from the plane parallel layers, there may be a first order effect in the eigenfunctions. This introduces phase distortion [13], and consequently anisotropy in the phase velocities as a first-order effect [8].

The fact that the lowest order solution is $k_0 = \sqrt{TS}$, is well known as the "adiabatic theorem" [14], for a system that undergoes infinitesimally slow changes. We have provided the exact correction to this result if the rate of transition is taken into account, for this particular mode of wave propagation. We anticipate that other systems such as quantum mechanical ones undergoing slow, finite transitions can be treated similarly.

Appendix I

We denote by $e^{-i\omega t}\mathbf{\Phi}^0(\mathbf{x})$ the displacement vector due to the incident waves. Then

$$\begin{aligned}\mathbf{\Phi}^0(\mathbf{x}) &= \mathbf{u}^0(\mathbf{x}),\ 0 < x_3 < H\\ &= \mathbf{v}^0(\mathbf{x}),\ x_3 < 0.\end{aligned} \qquad \text{(AI. 1)}$$

Let $e^{-i\omega t}\mathbf{\Phi}(\mathbf{x})$ and $e^{-i\omega t}\mathbf{\Psi}(\mathbf{x})$ be the displacement vectors in the regions $R_1 + R_2$ and R_3, respectively, satisfying the boundary conditions on S_0, $x_3 = -\infty$ and the continuity conditions across the surfaces S, S_1, S_2. Consider the expression

$$\{G_{ki}(\boldsymbol{\xi}, \mathbf{x})\, \tau_{ij}(\mathbf{\Phi}) - \Phi_i(\boldsymbol{\xi})\, \tau_{ij}(\mathbf{G}_k)\}\, n_j\,, \qquad \text{(AI. 2)}$$

where $\boldsymbol{\xi}$ is a point on any surface outside the region R_3, and n_j is the outward drawn unit normal vector at $\boldsymbol{\xi}$. Because of the continuity of the displacements and the normal stresses, the above expression is continuous everywhere including the boundary surfaces. Let $\mathbf{x}$ be a point in the region $R_1 + R_2$ bounded internally by the surfaces S_1 and S_2 and externally by the rectangle formed by the free surface S_0, a parallel line at $x_3 = -\infty$ and the two vertical lines $x_1 = \pm\infty$. For all points $\mathbf{x} \in R_1 + R_2$, the Green's function $\mathbf{G}_k(\boldsymbol{\xi}, \mathbf{x})$ and the displacement vectors $\mathbf{\Phi}(\mathbf{x})$ and $\mathbf{\Phi}^0(\mathbf{x})$ satisfy the same differential equations and boundary conditions except at the point $\boldsymbol{\xi} = \mathbf{x}$ where the Green's function is singular. We further assume that the Green's function is normalized [1].

We integrate expression (AI. 2) along the boundary of $R_1 + R_2$ and by usual procedures obtain

$$\Phi_\kappa(\mathbf{x}) = \Phi_\kappa{}^0(\mathbf{x}) - \int_{S_1+S_2} \{G_{\kappa i}(\boldsymbol{\xi}, \mathbf{x})\tau_{ij}(\mathbf{\Phi}) - \Phi_i(\boldsymbol{\xi})\tau_{ij}(\mathbf{G}_\kappa)\} n_j\, dS(\boldsymbol{\xi}), \qquad \text{(AI.3)}$$

where n_j is the outward normal with respect to the region R_3.

The boundary conditions on S_1 and S_2 are

$$\begin{aligned}\mathbf{\Phi}(\boldsymbol{\xi}) &= \mathbf{\Psi}(\boldsymbol{\xi})\\ \tau_{ij}(\mathbf{\Phi})\, n_j &= \sigma_{ij}(\mathbf{\Psi})\, n_j\,.\end{aligned} \qquad \text{(AI. 4)}$$

It is to be noted that in (AI. 3) the stresses τ_{ij} and σ_{ij} involve different elastic constants. We convert the surface integral in (AI. 3) into the volume integral

$$\int_{R_3} \{G_{\kappa i}(\boldsymbol{\xi}, \mathbf{x})\sigma_{ij}(\psi) - \Psi_i(\boldsymbol{\xi})\tau_{ij}(\mathbf{G}_\kappa)\}_{,j}\, d\boldsymbol{\xi}$$

and use the equations

$$\begin{aligned}\sigma_{ij,j}(\mathbf{\Psi}) + \rho_2\omega^2\Psi_i(\boldsymbol{\xi}) &= 0,\ \boldsymbol{\xi} \in R_3\\ \tau_{ij,j}(\mathbf{G}_k) + \rho_1\omega^2 G_{ki}(\boldsymbol{\xi}, \mathbf{x}) &= 0,\ \boldsymbol{\xi} \in R_1 + R_2\end{aligned}$$

to obtain Eq. (2).

APPENDIX II

We prove some useful results concerning the wave function due to free harmonic Rayleigh waves propagating in a multi-layered medium in which each layer is of uniform thickness.

(i) It is well known that the horizontal and the vertical components of the displacement function for two dimensional Rayleigh waves propagating in a uniform half space differ in phase by $\pi/2$. This result is also true for multi-layered media. Let $U_1(x_3)\cos(kx_1 - \omega t)$, $U_3(x_3)\cos(kx_1 - \omega t + \delta)$ be the *real* displacement functions, where $x_3 = 0$ is the free surface. If λ_1 and μ_1 are the Lamé constants of the top layer, the boundary conditions at the free surface are

$$\left[(\lambda_1 + 2\mu_1)\frac{dU_3}{dx_3}\cos(kx_1 - \omega t + \delta) - \lambda_1 k U_1(x_3)\sin(kx_1 - \omega t)\right]_{x_3=0} = 0$$

$$\left[\frac{dU_1}{dx_3}\cos(kx_1 - \omega t) - kU_3(x_3)\sin(kx_1 - \omega t + \delta)\right]_{x_3=0} = 0.$$

The above two equations can hold at every point on the free surface only if $|\delta| = \pi/2$.

Thus in expressions (12) and (13) $\mathbf{u}^0(\mathbf{x})$ and $\mathbf{v}^0(\mathbf{x})$ can be written in the forms

$$\begin{aligned} \Phi_1{}^0(\mathbf{x}) &= i\Phi_{11}^0(x_3)e^{ik_0x_1} \\ \Phi_3{}^0(\mathbf{x}) &= \Phi_{13}^0(x_3)e^{ik_0x_1}, \end{aligned} \tag{AII. 1}$$

where $\Phi_{11}^0(x_3)$, $\Phi_{13}^0(x_3)$ are real functions of x_3.

(ii) The normalization factor R of the Green's function is given by [1]

$$R = -\int_{-\infty}^{H} \{\bar{\Phi}_i{}^0(\xi)\tau_{i1}^0(\mathbf{\Phi}) - \Phi_i{}^0(\xi)\bar{\tau}_{i1}^0(\mathbf{\Phi})\}\, d\xi_3\,.$$

Equation (7) follows from the fact that the two terms in the integrand for R are complex conjugates of one another.

Let λ, μ be the elastic constants at any point ξ. Then,

$$\bar{\Phi}_i{}^0(\xi)\tau_{i1}^0(\mathbf{\Phi}) = \bar{\Phi}_i{}^0(\xi)\left\{\lambda\delta_{i1}\left(\frac{\partial\Phi_3{}^0}{\partial\xi_3} + ik_0\Phi_1{}^0\right) + \mu\left(ik_0\Phi_i{}^0 + \frac{\partial\Phi_1{}^0}{\partial\xi_i}\right)\right\}.$$

By using (AII. 1) the above expression can be easily shown to be purely imaginary. Hence we can write

$$R = iR_0\,, \tag{AII. 2}$$

where R_0 is real.

(iii) Consider the scalar quantity

$$c_{ijpq}\bar{u}^0_{i,j}v^0_{p,q} ,$$

where

$$c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}).$$

$\mathbf{u}^0(\boldsymbol{\xi})$ and $\mathbf{v}^0(\boldsymbol{\xi})$ have the forms given by (AII. 1). By direct calculation it can be easily shown that the above expression is a real function of ξ_3. The product $\bar{u}_i{}^0(\boldsymbol{\xi})\, v_i{}^0(\boldsymbol{\xi})$ is also real and independent of ξ_1. Thus in (12) we may write

$$P_0(\xi_1) = P_1(h)$$

$$Q_0(\xi_1) = Q_1(h)\, e^{2ik_0\xi_1} , \qquad \text{(AII. 3)}$$

where $P_1(h)$ and $Q_1(h)$ are real functions of $h(\xi_1)$ only.

(iv) By a similar argument Eqs. (15) can be shown to be equivalent to

$$A_0(\xi_1) = iA_1(h)$$

$$B_0(\xi_1) = iB_1(h)\, e^{2ik_0\xi_1} , \qquad \text{(AII. 4)}$$

where $A_1(h)$ and $B_1(h)$ are real functions of h.

References

1. I. Herrera. A perturbation method for elastic wave propagation 1—nonparallel boundaries. *J. Geophys. Res.* **69** (1964), 3845.
2. A. K. Mal and I. Herrera. Scattering of Love Waves by a constriction in the crust. *J. Geophys. Res.* **70** (1965), 5125.
3. L. Knopoff and J. A. Hudson. Transmission of Love Waves past a continental margin. *J. Geophys. Res.* **69** (1964), 1649.
4. A. K. Mal and L. Knopoff. Transmission of Rayleigh waves past a step change in elevation. *Bull. Seis. Soc. Amer.* **55** (1965), 319.
5. J. DeNoyer. The effect of variation in layer thickness on Love waves. *Bull. Seis. Soc. Amer.* **51** (1961), 227.
6. J. Kuo and J. Nafe. Period equation of Rayleigh waves in a layer overlying a half-space with a sinusoidal interface. *Bull. Seis. Soc. Amer.* **52** (1962), 807.
7. J. Kuo and G. A. Thompson. Model studies on the effect of a sloping interface on Rayleigh waves. *J. Geophys. Res.* **68** (1963), 6187.
8. L. Knopoff and A. K. Mal. The phase velocity of surface waves in the transitional zone of continental margins. *J. Geophys. Res.* **72** (1967), 1769.
9. R. Bellman and R. Kalaba. Functional equations, wave propagation and invariant imbedding. *J. Math. Mech.* **8** (1959), 683.
10. A. K. Mal and L. Knopoff. Elastic wave velocities in two-component systems. *J. Inst. Math. Appl.* (1968), in press.

11. H. Bremmer. The W.K.B. approximation as the first term of a geometric optical series. *Comm. Pure Appl. Math.* **4** (1951), 105.
12. S. A. Schelkunoff. Remarks concerning wave propagation in stratified media. *Comm. Pure. Appl. Math.* 4 (1951), 117.
13. J. Kane. Travel time and phase shift. *J. Geophys. Res.* **70** (1965), 1893.
14. K. O. Friedrichs. Special topics in analysis. New York University, Notes by Friedrichs and Kranzer. (1953-54), B-201.